Collection of formulae for the Euler constant

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1 Integral formulae

Euler's constant may be founded in many integrals, we provide a selection of such integrals. Some of those can be deduced from others by a simple change of variables.

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\begin{aligned}
\gamma &= -\int_0^\infty e^{-t}\log(t)\,dt = -\Gamma'(1) \\
\gamma^2 + \frac{\pi^2}{6} &= \int_0^\infty e^{-t}\log^2(t)\,dt = \Gamma''(1) \qquad (\mathrm{Euler-MascheroniIntegral}) \\
\gamma &= -\int_0^1 \log(\log(1/t))dt \\
\gamma + 2\log(2) &= -4\,\pi^{-1/2}\int_0^\infty e^{-t^2}\,\log(t)\,dt \\
\gamma &= \int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt \\
\gamma &= \int_0^1 \left(\frac{1}{t} + \frac{1}{\log(1-t)}\right)dt \\
\gamma &= \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right)\frac{dt}{t} \\
\gamma &= \int_0^\infty \left(\frac{1}{1+t^2} - \cos(t)\right)\frac{dt}{t} \\
(\alpha-\beta)\gamma &= \alpha\beta\int_0^\infty \frac{e^{-t^\alpha} - e^{-t^\beta}}{t}dt \qquad \alpha>0, \beta>0 \\
\gamma &= 1 - \int_0^1 \frac{1}{1+t}\left(\sum_{k=1}^\infty t^{2^k}\right)dt \qquad (\mathrm{Catalan's formula}) \\
\gamma &= \frac{1}{2} + 2\int_0^\infty \frac{t\,dt}{(t^2+1)(e^{2\pi t}-1)} \qquad (\mathrm{Hermite's formula}) \\
\gamma &= 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \log(n) + 2\int_0^\infty \frac{t\,dt}{(t^2+n^2)(e^{2\pi t}-1)} \\
\gamma &= \int_0^1 \frac{1 - e^{-t} - e^{-1/t}}{t}\,dt \qquad (\mathrm{Barnes})
\end{aligned}
$$

[1] This pages are from *//xavier.gourdon.free.fr/Constants/constants.html*

$$\gamma = \int_0^x \frac{1-\cos(t)}{t}dt - \int_x^\infty \frac{\cos(t)}{t}dt - \log(x) \qquad x>0$$
$$\gamma = \int_0^x \frac{1-e^{-t}}{t}dt - \int_x^\infty \frac{e^{-t}}{t}dt - \log(x) \qquad x>0$$

The last integral is often used to compute many digits of γ.

2 Series formulae

Now here is a list of series for γ.

2.1 Basic series

$$\gamma = \lim_{m\to\infty}(\sum_{k=1}^m \frac{1}{k} - \log(m)) \qquad \text{(Euler)}$$
$$\gamma = 1+\sum_{k=2}^\infty \left[\frac{1}{k}+\log(1-\frac{1}{k})\right] \qquad \text{(Euler)}$$
$$\gamma = 2\left(1+\frac{1}{2}\frac{m-1}{m}+\frac{1}{3}\frac{m-1}{m}\frac{m-2}{m}+...\right)-\log(2m) \qquad \text{(Kruskal)}$$
$$\gamma = \lim_{n\to\infty}\left(n-\Gamma\left(\frac{1}{n}\right)\right) \qquad \text{(Demys)}$$
$$\gamma = \frac{\log(2)}{2}+\frac{1}{\log(2)}\sum_{k=2}^m(-1)^k\frac{\log(k)}{k}$$

2.2 Around the zeta function

When he studied γ, Euler found some interesting series which allow to compute it with the integral values of the Riemann zeta function. He used one of those to give the first estimation of his constant (a five digits approximation).

There are many formulae giving γ as function of the Riemann zeta function $\zeta(s)$, some are easy to prove. We provide the demonstration of one example.

By definition, we may write:

$$\gamma = \lim_{m\to\infty}(\sum_{k=1}^m \frac{1}{k}-\log(m)) = 1+\sum_{k=2}^\infty \left[\frac{1}{k}+\log(\frac{k-1}{k})\right] = 1+\sum_{k=2}^\infty \left[\frac{1}{k}+\log(1-\frac{1}{k})\right]$$

and using the series for $\log(1-x)$ when $x=1/k<1$:

$$\gamma = 1-\sum_{k=2}^\infty\left(\sum_{i=2}^\infty \frac{1}{ik^i}\right)$$

then by associativity of this positive sum:

$$\gamma = 1 - \sum_{i=2}^{\infty} \frac{1}{i} \left(\sum_{k=2}^{\infty} \frac{1}{k^i} \right) = 1 - \sum_{i=2}^{\infty} \frac{1}{i} \left(\zeta(i) - 1 \right)$$

So we've just demonstrated a first relation between γ and the zeta functions. Because it's clear that $\zeta(i) - 1$ is equivalent to $1/2^i$ when i becomes large, some of those series have geometric convergence (of course one has to evaluate $\zeta(i)$ for different values of i).

$$\begin{aligned}
\gamma &= 1 - \sum_{k=2}^{\infty} \frac{(\zeta(k) - 1)}{k} \qquad \text{(Euler)} \\
\gamma &= \sum_{k=2}^{\infty} \frac{(k-1)(\zeta(k) - 1)}{k} \qquad \text{(Euler)} \\
\gamma &= 1 - \frac{\log(2)}{2} - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1) - 1)}{2k+1} \\
\gamma &= \log(2) - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{4^k(2k+1)} \qquad \text{(Euler)} \\
\gamma &= 1 - \log(\frac{3}{2}) - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1) - 1)}{4^k(2k+1)} \qquad \text{(Euler} - \text{Stieltjes)} \\
\gamma &= 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)} \qquad \text{(Glaisher)} \\
\gamma &= \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} \qquad \text{(Euler)} \\
\gamma &= 1 - \log(2) + \sum_{k=2}^{\infty} (-1)^k \frac{(\zeta(k) - 1)}{k} \\
\gamma &= \frac{3}{2} - \log(2) - \sum_{k=2}^{\infty} (-1)^k (k-1) \frac{(\zeta(k) - 1)}{k} \qquad \text{(Flajolet} - \text{Vardi)} \\
\gamma &= \frac{5}{4} - \log(2) - \frac{1}{2} \sum_{k=3}^{\infty} (-1)^k (k-2) \frac{(\zeta(k) - 1)}{k} \\
\gamma &= \log(\frac{4}{\pi}) + 2 \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{2^k k} \\
\gamma &= 1 + \log(\frac{16}{9\pi}) + 2 \sum_{k=2}^{\infty} (-1)^k \frac{(\zeta(k) - 1)}{2^k k}
\end{aligned}$$

2.3 Other series

$$\gamma = \sum_{i=1}^{\infty} i \left(\sum_{j=2^i}^{j=2^{i+1}-1} \frac{(-1)^j}{j} \right) \qquad \text{(Vacca)}$$

$$\gamma = \sum_{k=1}^{\infty} \frac{a_k}{k} \qquad \text{(Kluyver)}$$

$$\gamma = 1 - \log(2) + \sum_{k=1}^{\infty} \frac{a_k}{k(k+1)} \qquad \text{(Kluyver)}$$

$$\gamma = \sum_{k=1}^{m-1} \frac{1}{k} - \log(m) + \frac{1}{m} \sum_{k=1}^{\infty} \left(\frac{a_k}{\binom{m+k-1}{m}} \right) \qquad \text{(Kluyver[1])}$$

in the last formulae the a_k are rational numbers defined by :

$$a_1 = \frac{1}{2}$$

$$a_k = \frac{1}{k+1} \sum_{i=1}^{k-1} \frac{k-i}{i(i+1)} a_{k-i}$$

and $0 < a_k \leq \frac{1}{k+1}$. We give the first values:

$$a_1 = \frac{1}{2}, a_2 = \frac{1}{12}, a_3 = \frac{1}{24}, a_4 = \frac{19}{720}, a_5 = \frac{3}{160}, a_6 = \frac{863}{60480}, a_7 = \frac{275}{24192} \ldots$$

Kluyver's last relation may be used to compute a few hundred digits of γ.

3 Euler's constant and the prime numbers

3.0.1 Mertens formulae

If p represents a prime number, Mertens gave in 1874 the two beautiful formulae [2]:

$$e^{\gamma} = \lim_{m \to \infty} \frac{1}{\log(m)} \prod_{p \leq m} (1 - \frac{1}{p})^{-1}$$

$$\frac{6e^{\gamma}}{\pi^2} = \lim_{m \to \infty} \frac{1}{\log(m)} \prod_{p \leq m} (1 + \frac{1}{p})$$

The first relation is equivalent to:

$$\gamma = \lim_{m \to \infty} \left(\sum_{p \leq m} - \log(1 - \frac{1}{p}) - \log(\log(m)) \right)$$

when p is large:

$$-\log(1-\frac{1}{p})=\frac{1}{p}+O(\frac{1}{p^2})$$

and the last relation for γ is very similar to it's definition relation, but this time, only the prime numbers are taken into account in the sum.

3.0.2 Euler's constant and von Mangoldt function

$$\gamma=-\frac{1}{2}\sum_{k=2}^{\infty}\frac{(\Lambda(k)-1)}{k}$$

$\Lambda(k)$ is *von Mangoldt function* and is defined by

$$\begin{aligned}\Lambda(k) = &\quad \log(p) \quad \text{if } k=p^m \text{ foranyprime } p\\ = &\quad 0 \quad \text{otherwise,}\end{aligned}$$

it is related to the logarithmic derivative of the zeta function by the interesting and similar relation

$$-\frac{\zeta^{'}(s)}{\zeta(s)}=\sum_{k=1}^{\infty}\frac{\Lambda(k)}{k^s} \qquad s>1.$$

References

[1] J. C. Kluyver, *De constante van Euler en de natuurlijhe getallen*, Amst. Ak., (1924), vol. 33, p. 149-151

[2] F. Mertens, Journal für Math., (1874), vol. 78, p. 46-62

[3] T. J. Stieltjes, *Tables des valeurs des sommes* $S_k=\sum_{n=1}^{\infty}n^{-k}$,Acta Mathematica, (1887), vol. 10, p. 299-302